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Definition of functionals of the geopotential used in GrafLab software

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Notation

Symbol	Definition
r, φ, λ	Spherical radius, spherical latitude and spherical longitude, respectively
n, m	Spherical harmonic degree and order, respectively
$n_{\min}$, $n_{\max}$	Minimum and maximum degree of the spherical harmonic expansion, respectively
$\bar{P}_{n,m}(\sin\varphi)$	4π fully normalized associated Legendre function of the first kind of degree n and order m
$\bar{C}_{n,m}$, $\bar{S}_{n,m}$	4π fully normalized spherical harmonic coefficients of degree n and order m related to global geopotential model
GM, R	Geocentric gravitational constant and radius of the reference sphere, respectively (scaling parameters of the coefficients $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$)
$\bar{C}^{\mathrm{Ell}}_{n,m}$, $\bar{S}^{\mathrm{Ell}}_{n,m}$	4π fully normalized spherical harmonic coefficients of degree n and order m related to the reference ellipsoid
GM^{Ell}, a^{Ell}	Geocentric gravitational constant and semimajor axis of the reference ellipsoid, respectively (scaling parameters of the coefficients $\bar{C}^{\mathrm{Ell}}_{n,m}$ and $\bar{S}^{\mathrm{Ell}}_{n,m}$)
e^2	Squared first eccentricity of the reference ellipsoid
ω	Angular velocity of the Earth

Gravitational potential

$$V(r,\varphi,\lambda) = \frac{GM}{r} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n \sum_{m=0}^{n} \left(\bar{C}_{n,m}\,\cos m\lambda + \bar{S}_{n,m}\,\sin m\lambda\right) \bar{P}_{n,m}(\sin\varphi) \tag{1}$$

Gravitational tensor in spherical coordinates

$$\mathbf{V}(r,\varphi,\lambda) = \begin{pmatrix} V_{rr} & V_{r\varphi} & V_{r\lambda} \\ V_{\varphi r} & V_{\varphi\varphi} & V_{\varphi\lambda} \\ V_{\lambda r} & V_{\lambda\varphi} & V_{\lambda\lambda} \end{pmatrix} \tag{2}$$

$$\begin{aligned} V_{rr}(r,\varphi,\lambda) &= \frac{\partial^2 V(r,\varphi,\lambda)}{\partial r^2} \\ &= \frac{GM}{r^3} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n (n+1)(n+2) \sum_{m=0}^{n} \left(\bar{C}_{n,m}\,\cos m\lambda + \bar{S}_{n,m}\,\sin m\lambda\right) \bar{P}_{n,m}(\sin\varphi) \end{aligned} \tag{3}$$

$$\begin{aligned} V_{r\varphi}(r,\varphi,\lambda) &= \frac{1}{r}\,\frac{\partial^2 V(r,\varphi,\lambda)}{\partial r\,\partial\varphi} \\ &= -\frac{GM}{r^3} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n (n+1) \sum_{m=0}^{n} \left(\bar{C}_{n,m}\,\cos m\lambda + \bar{S}_{n,m}\,\sin m\lambda\right) \frac{\mathrm{d}\bar{P}_{n,m}(\sin\varphi)}{\mathrm{d}\varphi} \end{aligned} \tag{4}$$

$$\begin{aligned} V_{r\lambda}(r,\varphi,\lambda) &= \frac{1}{r\cos\varphi}\,\frac{\partial^2 V(r,\varphi,\lambda)}{\partial r\,\partial\lambda} \\ &= -\frac{GM}{r^3\cos\varphi} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n (n+1) \sum_{m=0}^{n} \left(\bar{S}_{n,m}\,\cos m\lambda - \bar{C}_{n,m}\,\sin m\lambda\right) m\,\bar{P}_{n,m}(\sin\varphi) \end{aligned} \tag{5}$$

$$V_{\varphi\varphi}(r,\varphi,\lambda) = \frac{1}{r^2}\,\frac{\partial^2 V(r,\varphi,\lambda)}{\partial\varphi^2}$$
$$= \frac{GM}{r^3}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\left(\bar{C}_{n,m}\,\cos m\lambda + \bar{S}_{n,m}\,\sin m\lambda\right)\frac{\mathrm{d}^2\bar{P}_{n,m}(\sin\varphi)}{\mathrm{d}\varphi^2} \tag{6}$$

$$V_{\varphi\lambda}(r,\varphi,\lambda) = \frac{1}{r^2\cos\varphi}\,\frac{\partial^2 V(r,\varphi,\lambda)}{\partial\varphi\,\partial\lambda}$$
$$= \frac{GM}{r^3\cos\varphi}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\left(\bar{S}_{n,m}\,\cos m\lambda - \bar{C}_{n,m}\,\sin m\lambda\right) m\,\frac{\mathrm{d}\bar{P}_{n,m}(\sin\varphi)}{\mathrm{d}\varphi} \tag{7}$$

$$V_{\lambda\lambda}(r,\varphi,\lambda) = \frac{1}{r^2\cos^2\varphi}\,\frac{\partial^2 V(r,\varphi,\lambda)}{\partial\lambda^2}$$
$$= -\frac{GM}{r^3\cos^2\varphi}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\left(\bar{C}_{n,m}\,\cos m\lambda + \bar{S}_{n,m}\,\sin m\lambda\right) m^2\,\bar{P}_{n,m}(\sin\varphi) \tag{8}$$

Gravitational tensor in the local north-oriented reference frame[2]

$$\mathbf{V}(r,\varphi,\lambda) = \begin{pmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{pmatrix} \tag{9}$$

$$V_{xx}(r,\varphi,\lambda) = \frac{GM}{r^3}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n \sum_{m=-n}^{n}\bar{C}_{n,m}\,Q_m(\lambda)\Big(a_{n,m}\,\bar{P}_{n,|m|-2}(\sin\varphi) + \left[b_{n,m} - (n+1)(n+2)\right]\,\bar{P}_{n,|m|}(\sin\varphi) + c_{n,m}\,\bar{P}_{n,|m|+2}(\sin\varphi)\Big) \tag{10}$$

$$V_{xy}(r,\varphi,\lambda) = \frac{GM}{r^3}\sum_{n=0}^{n_{\max}}\left(\frac{R}{r}\right)^n \sum_{m=-n}^{n}\bar{C}_{n,m}\,Q_{-m}(\lambda)\Big(d_{n,m}\,\bar{P}_{n-1,|m|-2}(\sin\varphi) + g_{n,m}\,\bar{P}_{n-1,|m|}(\sin\varphi) + h_{n,m}\,\bar{P}_{n-1,|m|+2}(\sin\varphi)\Big),\quad m \neq 0 \tag{11}$$

$$V_{xz}(r,\varphi,\lambda) = \frac{GM}{r^3}\sum_{n=0}^{n_{\max}}\left(\frac{R}{r}\right)^n \sum_{m=-n}^{n}\bar{C}_{n,m}\,Q_m(\lambda)\Big(\beta_{n,m}\,\bar{P}_{n,|m|-1}(\sin\varphi) + \gamma_{n,m}\,\bar{P}_{n,|m|+1}(\sin\varphi)\Big) \tag{12}$$

[2] In GrafLab, Eqs. (10) – (15) have been slightly modified, see Appendix A.

$$V_{yy}(r,\varphi,\lambda) = -\frac{GM}{r^3}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n\sum_{m=-n}^{n}\bar{C}_{n,m}\,Q_m(\lambda)\Big(a_{n,m}\,\bar{P}_{n,|m|-2}(\sin\varphi) + b_{n,m}\,\bar{P}_{n,|m|}(\sin\varphi) + c_{n,m}\,\bar{P}_{n,|m|+2}(\sin\varphi)\Big) \tag{13}$$

$$V_{yz}(r,\varphi,\lambda) = \frac{GM}{r^3}\sum_{n=0}^{n_{\max}}\left(\frac{R}{r}\right)^n\sum_{m=-n}^{n}\bar{C}_{n,m}\,Q_{-m}(\lambda)\Big(\mu_{n,m}\,\bar{P}_{n-1,|m|-1}(\sin\varphi) + \nu_{n,m}\,\bar{P}_{n-1,|m|+1}(\sin\varphi)\Big), \quad m \neq 0 \tag{14}$$

$$V_{zz}(r,\varphi,\lambda) = \frac{GM}{r^3}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n (n+1)(n+2)\sum_{m=-n}^{n}\bar{C}_{n,m}\,Q_m(\lambda)\,\bar{P}_{n,|m|}(\sin\varphi) \tag{15}$$

$$Q_m(\lambda) = \begin{cases}\cos m\lambda, & m \geq 0\\ \sin |m|\lambda, & m < 0\end{cases} \tag{16}$$

$$a_{n,m} = 0, \quad |m| = 0, 1 \tag{17}$$

$$a_{n,m} = \frac{\sqrt{1+\delta_{|m|,2}}}{4}\sqrt{n^2-(|m|-1)^2}\sqrt{n+|m|}\sqrt{n-|m|+2}, \quad 2 \leq |m| \leq n \tag{18}$$

$$b_{n,m} = \frac{(n+|m|+1)(n+|m|+2)}{2\,(|m|+1)}, \quad |m| = 0, 1 \tag{19}$$

$$b_{n,m} = \frac{n^2+m^2+3n+2}{2}, \quad 2 \leq |m| \leq n \tag{20}$$

$$c_{n,m} = \frac{\sqrt{1+\delta_{|m|,0}}}{4}\sqrt{n^2-(|m|+1)^2}\sqrt{n-|m|}\sqrt{n+|m|+2}, \quad |m| = 0, 1 \tag{21}$$

$$c_{n,m} = \frac{1}{4}\sqrt{n^2-(|m|+1)^2}\sqrt{n-|m|}\sqrt{n+|m|+2}, \quad 2 \leq |m| \leq n \tag{22}$$

$$d_{n,m} = 0, \quad |m| = 1 \tag{23}$$

$$d_{n,m} = -\frac{m}{4|m|}\sqrt{\frac{2n+1}{2n-1}}\sqrt{1+\delta_{|m|,2}}\sqrt{n^2-(|m|-1)^2} \\ \times \sqrt{n+|m|}\sqrt{n+|m|-2}, \quad 2 \leq |m| \leq n \tag{24}$$

$$g_{n,m} = \frac{m}{4|m|}\sqrt{\frac{2n+1}{2n-1}}\sqrt{n+1}\sqrt{n-1}(n+2), \quad |m| = 1 \tag{25}$$

$$g_{n,m} = \frac{m}{2}\sqrt{\frac{2n+1}{2n-1}}\sqrt{n+|m|}\sqrt{n-|m|}, \quad 2 \leq |m| \leq n \tag{26}$$

$$h_{n,m} = \frac{m}{4|m|}\sqrt{\frac{2n+1}{2n-1}}\sqrt{n-3}\sqrt{n-2}\sqrt{n-1}\sqrt{n+2}, \quad |m| = 1 \tag{27}$$

$$h_{n,m} = \frac{m}{4|m|}\sqrt{\frac{2n+1}{2n-1}}\sqrt{n^2-(|m|+1)^2}\sqrt{n-|m|}\sqrt{n-|m|-2}, \quad 2 \leq |m| \leq n \tag{28}$$

$$\beta_{n,m} = 0, \quad m = 0 \tag{29}$$

$$\beta_{n,m} = \frac{n+2}{2}\sqrt{1+\delta_{|m|,1}}\sqrt{n+|m|}\sqrt{n-|m|+1}, \quad 1 \leq |m| \leq n \tag{30}$$

$$\gamma_{n,m} = -(n+2)\sqrt{\frac{n(n+1)}{2}}, \quad m = 0 \tag{31}$$

$$\gamma_{n,m} = -\frac{n+2}{2}\sqrt{n-|m|}\sqrt{n+|m|+1}, \quad 1 \leq |m| \leq n \tag{32}$$

$$\mu_{n,m} = -\frac{m}{|m|}\left(\frac{n+2}{2}\right)\sqrt{\frac{2n+1}{2n-1}}\sqrt{1+\delta_{|m|,1}}\sqrt{n+|m|}\sqrt{n+|m|-1} \tag{33}$$

$$\nu_{n,m} = -\frac{m}{|m|}\left(\frac{n+2}{2}\right)\sqrt{\frac{2n+1}{2n-1}}\sqrt{n-|m|}\sqrt{n-|m|-1} \tag{34}$$

$$\delta_{p,q} = \begin{cases} 1, & p = q, \\ 0, & p \neq q. \end{cases} \tag{35}$$

Gravity potential

$$W(r,\varphi,\lambda) = \frac{GM}{r} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n \sum_{m=0}^{n} \left(\bar{C}_{n,m}\,\cos m\lambda + \bar{S}_{n,m}\,\sin m\lambda\right) \bar{P}_{n,m}(\sin\varphi) + \frac{1}{2}\,\omega^2\, r^2\,\cos^2\varphi \tag{36}$$

Gravity vector in the local north-oriented reference frame

$$\mathbf{g}(r,\varphi,\lambda) = \nabla W(r,\varphi,\lambda) = \begin{bmatrix} g_X \\ g_Y \\ g_Z \end{bmatrix} \tag{37}$$

$$g_X(r,\varphi,\lambda) = \frac{1}{r}\left(\frac{\partial V(r,\varphi,\lambda)}{\partial\varphi} + \frac{\partial V_c(r,\varphi)}{\partial\varphi}\right) \tag{38}$$

$$g_Y(r,\varphi,\lambda) = -\frac{1}{r\cos\varphi}\left(\frac{\partial V(r,\varphi,\lambda)}{\partial\lambda} + \frac{\partial V_c(r,\varphi)}{\partial\lambda}\right) \tag{39}$$

$$g_Z(r,\varphi,\lambda) = \frac{\partial V(r,\varphi,\lambda)}{\partial r} + \frac{\partial V_c(r,\varphi)}{\partial r} \tag{40}$$

$$\begin{aligned}
\frac{\partial V(r,\varphi,\lambda)}{\partial r} &= -\frac{GM}{r^2} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n (n+1) \sum_{m=0}^{n} \left(\bar{C}_{n,m}\,\cos m\lambda + \bar{S}_{n,m}\,\sin m\lambda\right) \bar{P}_{n,m}(\sin\varphi) \\
\frac{\partial V(r,\varphi,\lambda)}{\partial\varphi} &= \frac{GM}{r} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n \sum_{m=0}^{n} \left(\bar{C}_{n,m}\,\cos m\lambda + \bar{S}_{n,m}\,\sin m\lambda\right) \frac{\mathrm{d}\bar{P}_{n,m}(\sin\varphi)}{\mathrm{d}\varphi} \\
\frac{\partial V(r,\varphi,\lambda)}{\partial\lambda} &= \frac{GM}{r} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n \sum_{m=0}^{n} \left(\bar{S}_{n,m}\,\cos m\lambda - \bar{C}_{n,m}\,\sin m\lambda\right) m\,\bar{P}_{n,m}(\sin\varphi)
\end{aligned} \tag{41}$$

$$V_c(r,\varphi) = \frac{1}{2}\,\omega^2\, r^2\,\cos^2\varphi \tag{42}$$

$$\frac{\partial V_c(r,\varphi)}{\partial r} = \omega^2\, r\,\cos^2\varphi, \quad \frac{\partial V_c(r,\varphi)}{\partial\varphi} = -\omega^2\, r^2\,\cos\varphi\,\sin\varphi, \quad \frac{\partial V_c(r,\varphi)}{\partial\lambda} = 0 \tag{43}$$

The magnitude of the gravity vector can be computed as

$$g(r,\varphi,\lambda) = |\mathbf{g}(r,\varphi,\lambda)| = \sqrt{g_X^2(r,\varphi,\lambda) + g_Y^2(r,\varphi,\lambda) + g_Z^2(r,\varphi,\lambda)} \tag{44}$$

Gravity sa (spherical approximation)

$$g_{\mathrm{sa}}(r,\varphi,\lambda) = \sqrt{\left(\frac{\partial V}{\partial r} + \frac{\partial V_c}{\partial r}\right)^2} \tag{45}$$

Second radial derivative of gravity potential

$$W_{rr}(r,\varphi,\lambda) = \frac{\partial^2 W(r,\varphi,\lambda)}{\partial r^2} = \frac{\partial^2 V(r,\varphi,\lambda)}{\partial r^2} + \frac{\partial^2 V_c(r,\varphi,\lambda)}{\partial r^2} \tag{46}$$

$$\begin{aligned} &\frac{\partial^2 V(r,\varphi,\lambda)}{\partial r^2} \\ &= \frac{GM}{r^3} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n (n+1)(n+2) \sum_{m=0}^{n} \left(\bar{C}_{n,m}\,\cos m\lambda + \bar{S}_{n,m}\,\sin m\lambda\right) \bar{P}_{n,m}(\sin\varphi) \end{aligned} \tag{47}$$

$$\frac{\partial^2 V_c(r,\varphi,\lambda)}{\partial r^2} = \omega^2\,\cos^2\varphi \tag{48}$$

Disturbing potential

$$T(r,\varphi,\lambda) = \frac{GM}{r} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n \sum_{m=0}^{n} \left(\Delta\bar{C}_{n,m}\,\cos m\lambda + \Delta\bar{S}_{n,m}\,\sin m\lambda\right) \bar{P}_{n,m}(\sin\varphi) \tag{49}$$

$$\Delta\bar{C}_{n,m} = \bar{C}_{n,m} - \bar{C}^{\mathrm{Ell}}_{n,m}\,\frac{GM^{\mathrm{Ell}}}{GM}\left(\frac{a^{\mathrm{Ell}}}{R}\right)^n \tag{50}$$

$$\Delta\bar{S}_{n,m} = \bar{S}_{n,m} - \bar{S}^{\mathrm{Ell}}_{n,m}\,\frac{GM^{\mathrm{Ell}}}{GM}\left(\frac{a^{\mathrm{Ell}}}{R}\right)^n = \bar{S}_{n,m} \tag{51}$$

$$\bar{C}^{\mathrm{Ell}}_{2n,m} = \begin{cases} (-1)^n\,\frac{3e^{2n}}{(2n+1)(2n+3)\sqrt{4n+1}}\left(1 - n - 5^{3/2}\,n\,\frac{\bar{C}^{\mathrm{Ell}}_{2,0}}{e^2}\right) & \text{if} \quad n = 0,1,2,\ldots,10,\ m=0 \\ 0 \quad \text{else} \end{cases} \tag{52}$$

$$\bar{S}^{\mathrm{Ell}}_{n,m} = 0 \quad \text{for all} \quad n,\,m \tag{53}$$

Gravity disturbance

$$\delta g(r,\varphi,\lambda) = g(r,\varphi,\lambda) - \gamma_{SH}(r,\varphi) \tag{54}$$

- $\gamma_{SH}(r,\varphi)$ is the normal gravity evaluated from its spherical harmonic expansion. The same formula as for $g(r,\varphi,\lambda)$ (see Eq. 44) holds for $\gamma_{SH}(r,\varphi)$, but instead of the coefficients $\bar{C}_{n,m}$, $\bar{S}_{n,m}$, ones employs in Eq. (41) the coefficients $\bar{C}^{\mathrm{Ell}}_{n,m}$, $\bar{S}^{\mathrm{Ell}}_{n,m}$.

Gravity disturbance sa (spherical approximation)

$$\delta g_{\mathrm{sa}}(r,\varphi,\lambda) = -\frac{\partial T(r,\varphi,\lambda)}{\partial r}$$
$$= \frac{GM}{r^2} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n (n+1) \sum_{m=0}^{n} \left(\Delta\bar{C}_{n,m}\, \cos m\lambda + \Delta\bar{S}_{n,m}\, \sin m\lambda\right) \bar{P}_{n,m}(\sin\varphi) \tag{55}$$

Gravity anomaly sa (spherical approximation)

$$\Delta g_{\mathrm{sa}}(r,\varphi,\lambda) = -\frac{\partial T(r,\varphi,\lambda)}{\partial r} - \frac{2}{r} T(r,\varphi,\lambda)$$
$$= \frac{GM}{r^2} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n (n-1) \sum_{m=0}^{n} \left(\Delta\bar{C}_{n,m}\, \cos m\lambda + \Delta\bar{S}_{n,m}\, \sin m\lambda\right) \bar{P}_{n,m}(\sin\varphi) \tag{56}$$

Second radial derivative of disturbing potential

$$T_{rr}(r,\varphi,\lambda) = \frac{\partial^2 T(r,\varphi,\lambda)}{\partial r^2}$$
$$= \frac{GM}{r^3} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n (n+1)(n+2) \sum_{m=0}^{n} \left(\Delta\bar{C}_{n,m}\, \cos m\lambda + \Delta\bar{S}_{n,m}\, \sin m\lambda\right) \bar{P}_{n,m}(\sin\varphi) \tag{57}$$

Disturbing tensor in spherical coordinates

$$\mathbf{T}(r,\varphi,\lambda) = \begin{pmatrix} T_{rr} & T_{r\varphi} & T_{r\lambda} \\ T_{\varphi r} & T_{\varphi\varphi} & T_{\varphi\lambda} \\ T_{\lambda r} & T_{\lambda\varphi} & T_{\lambda\lambda} \end{pmatrix} \tag{58}$$

$$T_{rr}(r,\varphi,\lambda) = \frac{\partial^2 T(r,\varphi,\lambda)}{\partial r^2}$$
$$= \frac{GM}{r^3} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n (n+1)(n+2) \sum_{m=0}^{n} \left(\Delta\bar{C}_{n,m}\, \cos m\lambda + \Delta\bar{S}_{n,m}\, \sin m\lambda\right) \bar{P}_{n,m}(\sin\varphi) \tag{59}$$

$$T_{r\varphi}(r,\varphi,\lambda) = \frac{1}{r} \frac{\partial^2 T(r,\varphi,\lambda)}{\partial r\, \partial\varphi}$$
$$= -\frac{GM}{r^3} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n (n+1) \sum_{m=0}^{n} \left(\Delta\bar{C}_{n,m}\, \cos m\lambda + \Delta\bar{S}_{n,m}\, \sin m\lambda\right) \frac{\mathrm{d}\bar{P}_{n,m}(\sin\varphi)}{\mathrm{d}\varphi} \tag{60}$$

$$T_{r\lambda}(r,\varphi,\lambda) = \frac{1}{r\cos\varphi}\,\frac{\partial^2 T(r,\varphi,\lambda)}{\partial r\,\partial\lambda}$$
$$= -\frac{GM}{r^3\,\cos\varphi}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n (n+1)\sum_{m=0}^{n}\big(\Delta\bar{S}_{n,m}\,\cos m\lambda - \Delta\bar{C}_{n,m}\,\sin m\lambda\big) m\,\bar{P}_{n,m}(\sin\varphi) \tag{61}$$

$$\begin{aligned} T_{\varphi\varphi}(r,\varphi,\lambda) &= \frac{1}{r^2}\,\frac{\partial^2 T(r,\varphi,\lambda)}{\partial\varphi^2} \\ &= \frac{GM}{r^3}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\big(\Delta\bar{C}_{n,m}\,\cos m\lambda + \Delta\bar{S}_{n,m}\,\sin m\lambda\big)\,\frac{\mathrm{d}^2\bar{P}_{n,m}(\sin\varphi)}{\mathrm{d}\varphi^2} \end{aligned} \tag{62}$$

$$\begin{aligned} T_{\varphi\lambda}(r,\varphi,\lambda) &= \frac{1}{r^2\cos\varphi}\,\frac{\partial^2 T(r,\varphi,\lambda)}{\partial\varphi\,\partial\lambda} \\ &= \frac{GM}{r^3\,\cos\varphi}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\big(\Delta\bar{S}_{n,m}\,\cos m\lambda - \Delta\bar{C}_{n,m}\,\sin m\lambda\big)\, m\,\frac{\mathrm{d}\bar{P}_{n,m}(\sin\varphi)}{\mathrm{d}\varphi} \end{aligned} \tag{63}$$

$$\begin{aligned} T_{\lambda\lambda}(r,\varphi,\lambda) &= \frac{1}{r^2\cos^2\varphi}\,\frac{\partial^2 T(r,\varphi,\lambda)}{\partial\lambda^2} \\ &= -\frac{GM}{r^3\,\cos^2\varphi}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n}\big(\Delta\bar{C}_{n,m}\,\cos m\lambda + \Delta\bar{S}_{n,m}\,\sin m\lambda\big)\, m^2\,\bar{P}_{n,m}(\sin\varphi) \end{aligned} \tag{64}$$

Disturbing tensor in the local north-oriented reference frame[3]

$$\mathbf{T}(r,\varphi,\lambda) = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \tag{65}$$

$$\begin{aligned} T_{xx}(r,\varphi,\lambda) =& \frac{GM}{r^3}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n \sum_{m=-n}^{n}\Delta\bar{C}_{n,m}\,Q_m(\lambda)\Big(a_{n,m}\,\bar{P}_{n,|m|-2}(\sin\varphi) \\ &+ [b_{n,m} - (n+1)(n+2)]\;\bar{P}_{n,|m|}(\sin\varphi) + c_{n,m}\,\bar{P}_{n,|m|+2}(\sin\varphi)\Big) \end{aligned} \tag{66}$$

$$\begin{aligned} T_{xy}(r,\varphi,\lambda) =& \frac{GM}{r^3}\sum_{n=0}^{n_{\max}}\left(\frac{R}{r}\right)^n \sum_{m=-n}^{n}\Delta\bar{C}_{n,m}\,Q_{-m}(\lambda)\Big(d_{n,m}\,\bar{P}_{n-1,|m|-2}(\sin\varphi) \\ &+ g_{n,m}\,\bar{P}_{n-1,|m|}(\sin\varphi) + h_{n,m}\,\bar{P}_{n-1,|m|+2}(\sin\varphi)\Big), \quad m \neq 0 \end{aligned} \tag{67}$$

[3] In GrafLab, Eqs. (66) – (71) have been slightly modified, see Appendix A.

$$
\begin{aligned}
T_{xz}(r,\varphi,\lambda) =& \frac{GM}{r^3}\sum_{n=0}^{n_{\max}}\left(\frac{R}{r}\right)^n\sum_{m=-n}^{n}\Delta\bar{C}_{n,m}\,Q_m(\lambda)\Big(\beta_{n,m}\,\bar{P}_{n,|m|-1}(\sin\varphi) \\
&+\gamma_{n,m}\,\bar{P}_{n,|m|+1}(\sin\varphi)\Big)
\end{aligned} \tag{68}
$$

$$
\begin{aligned}
T_{yy}(r,\varphi,\lambda) =& -\frac{GM}{r^3}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n\sum_{m=-n}^{n}\Delta\bar{C}_{n,m}\,Q_m(\lambda)\Big(a_{n,m}\,\bar{P}_{n,|m|-2}(\sin\varphi) \\
&+b_{n,m}\,\bar{P}_{n,|m|}(\sin\varphi)+c_{n,m}\,\bar{P}_{n,|m|+2}(\sin\varphi)\Big)
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
T_{yz}(r,\varphi,\lambda) =& \frac{GM}{r^3}\sum_{n=0}^{n_{\max}}\left(\frac{R}{r}\right)^n\sum_{m=-n}^{n}\Delta\bar{C}_{n,m}\,Q_{-m}(\lambda)\Big(\mu_{n,m}\,\bar{P}_{n-1,|m|-1}(\sin\varphi) \\
&+\nu_{n,m}\,\bar{P}_{n-1,|m|+1}(\sin\varphi)\Big),\quad m\neq 0
\end{aligned} \tag{70}
$$

$$
T_{zz}(r,\varphi,\lambda) = \frac{GM}{r^3}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n(n+1)(n+2)\sum_{m=-n}^{n}\Delta\bar{C}_{n,m}\,Q_m(\lambda)\,\bar{P}_{n,|m|}(\sin\varphi) \tag{71}
$$

Deflections of the vertical

$$
\begin{aligned}
\xi(r,\varphi,\lambda) &= -\frac{1}{r\,\gamma(r,\varphi)}\frac{\partial T(r,\varphi,\lambda)}{\partial\varphi} \\
&= -\frac{GM}{r^2\,\gamma(r,\varphi)}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n\sum_{m=0}^{n}\left(\Delta\bar{C}_{n,m}\,\cos m\lambda+\Delta\bar{S}_{n,m}\,\sin m\lambda\right)\frac{\mathrm{d}\bar{P}_{n,m}(\sin\varphi)}{\mathrm{d}\varphi}
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
\eta(r,\varphi,\lambda) &= -\frac{1}{r\,\gamma(r,\varphi)\cos\varphi}\frac{\partial T(r,\varphi,\lambda)}{\partial\lambda} \\
&= -\frac{GM}{r^2\,\gamma(r,\varphi)\cos\varphi}\sum_{n=n_{\min}}^{n_{\max}}\left(\frac{R}{r}\right)^n\sum_{m=0}^{n}\left(\Delta\bar{S}_{n,m}\,\cos m\lambda-\Delta\bar{C}_{n,m}\,\sin m\lambda\right)m\,\bar{P}_{n,m}(\sin\varphi)
\end{aligned} \tag{73}
$$

$$
\Theta(r,\varphi,\lambda) = \sqrt{\xi^2(r,\varphi,\lambda)+\eta^2(r,\varphi,\lambda)} \tag{74}
$$

Geoid undulation

$$
H(\varphi,\lambda) = \sum_{n=0}^{n_{\max}}\sum_{m=0}^{n}\left(\overline{HC}_{n,m}\,\cos m\lambda+\overline{HS}_{n,m}\,\sin m\lambda\right)\bar{P}_{n,m}(\sin\varphi) \tag{75}
$$

$$
N(\varphi,\lambda) = \frac{T(r_{\mathrm{ell}},\varphi,\lambda)-2\pi\,G\,\rho\,H^2(\varphi,\lambda)}{\gamma(r_{\mathrm{ell}},\varphi)} \tag{76}
$$

- G denotes the Newtonian gravitational constant, $G = 6.67259 \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$ (Moritz, 2000)
- ρ denotes the density of the crust, $\rho = 2670\ \mathrm{kg\ m^{-3}}$
- $r_{\mathrm{ell}} = r_{\mathrm{ell}}(\varphi)$ denotes the spherical radius of the point lying on the reference ellipsoid

Height anomaly ell

$$\zeta_{\mathrm{ell}}(r, \varphi, \lambda) = \frac{T(r, \varphi, \lambda)}{\gamma(r, \varphi)} \tag{77}$$

Height anomaly

$$\zeta(r, \varphi, \lambda) = \zeta_{\mathrm{ell}}(r_{\mathrm{ell}}, \varphi, \lambda) - \delta g_{\mathrm{sa}}(r_{\mathrm{ell}}, \varphi, \lambda)\, \frac{H(\varphi, \lambda) + N(\varphi, \lambda)}{\gamma(r_{\mathrm{ell}}, \varphi)} \tag{78}$$

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A Modified non-singular expressions for the gravity gradients in the LNOF

In this appendix, we demonstrate a modification of the non-singular expressions for the gravity gradients in the LNOF (Eqs. (10) – (15) and Eqs. (66) – (71); see Petrovskaya and Vershkov 2006). As an example, we chose the element T_{xx}, which has the following form

$$
\begin{aligned}
T_{xx}(r,\varphi,\lambda) =& \frac{GM}{r^3} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n \sum_{m=0}^{n} \left(\bar{C}_{nm}\,\cos m\lambda + \bar{S}_{nm}\,\sin m\lambda\right) \\
& \times \left[a_{nm}\,\bar{P}_{n,m-2}(\sin\varphi) + \left[b_{nm} - (n+1)(n+2)\right]\,\bar{P}_{nm}(\sin\varphi) + c_{nm}\,\bar{P}_{n,m+2}(\sin\varphi)\right],
\end{aligned} \tag{79}
$$

in which

$$
a_{nm} = 0, \quad m = 0, 1 \tag{80}
$$

$$
\begin{aligned}
a_{nm} =& \frac{\sqrt{1+\delta_{m,2}}}{4} \sqrt{n^2 - (m-1)^2} \\
& \times \sqrt{n+m}\sqrt{n-m+2}, \quad 2 \le m \le n
\end{aligned} \tag{81}
$$

$$
b_{nm} = \frac{(n+m+1)(n+m+2)}{2\,(m+1)}, \quad m = 0, 1 \tag{82}
$$

$$
b_{nm} = \frac{n^2 + m^2 + 3n + 2}{2}, \quad 2 \le m \le n \tag{83}
$$

$$
\begin{aligned}
c_{nm} =& \frac{\sqrt{1+\delta_{m,0}}}{4} \sqrt{n^2 - (m+1)^2}\sqrt{n-m} \\
& \times \sqrt{n+m+2}, \quad m = 0, 1
\end{aligned} \tag{84}
$$

$$
c_{nm} = \frac{1}{4}\sqrt{n^2 - (m+1)^2}\sqrt{n-m}\sqrt{n+m+2}, \quad 2 \le m \le n \tag{85}
$$

$$
\delta_{p,q} = \begin{cases} 1, & p = q, \\ 0, & p \ne q. \end{cases} \tag{86}
$$

From Eq. (79) it is seen that in addition to $\bar{P}_{nm}(\sin\varphi)$, two other terms $\bar{P}_{n,m-2}(\sin\varphi)$ and $\bar{P}_{n,m+2}(\sin\varphi)$ need to be computed for each m. From the practical numerical point of view, this is not an issue if fixed-degree recursions are used to evaluate the fully normalized associated Legendre functions. In GrafLab, however, we implemented the fixed-order recursions, which are more frequently used in geodesy. In this case, with every change of m in the order-dependent loop, it is necessary to evaluate not only $\bar{P}_{nm}(\sin\varphi)$, but also the two other terms. In other words, redundant computations occur. Therefore, we modified Eq. (79) in such a way that only the term $\bar{P}_{nm}(\sin\varphi)$ is needed. We present Eq. (79) in the following form

$$\begin{aligned} T_{xx}(r,\varphi,\lambda) =& \frac{GM}{r^3} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^n \sum_{m=0}^{n} \Bigg[\left(\bar{C}_{n,m+2}\, \cos(m+2)\lambda + \bar{S}_{n,m+2}\, \sin(m+2)\lambda\right)\, a_{n,m+2} \\ & + \left(\bar{C}_{nm}\, \cos m\lambda + \bar{S}_{nm}\, \sin m\lambda\right)\, (b_{nm} - (n+1)(n+2)) \\ & + \left(\bar{C}_{n,m-2}\, \cos(m-2)\lambda + \bar{S}_{n,m-2}\, \sin(m-2)\lambda\right)\, c_{n,m-2} \Bigg] \\ & \times \bar{P}_{nm}(\sin\varphi), \end{aligned} \tag{87}$$

where

$$\left.\begin{array}{l} \bar{C}_{n,m+2} \\ \bar{S}_{n,m+2} \\ \cos(m+2)\lambda \\ \sin(m+2)\lambda \\ a_{n,m+2} \end{array}\right\} = 0, \quad m+2 > n, \tag{88}$$

$$\left.\begin{array}{l} \bar{C}_{n,m-2} \\ \bar{S}_{n,m-2} \\ \cos(m-2)\lambda \\ \sin(m-2)\lambda \\ c_{n,m-2} \end{array}\right\} = 0, \quad m-2 < 0. \tag{89}$$

The main idea of Eq. (87) is that the set of spherical harmonic coefficients is usually stored during the whole computational process, hence the coefficients $\bar{C}_{n,m+2}, \bar{S}_{n,m+2}$ and $\bar{C}_{n,m-2}, \bar{S}_{n,m-2}$ may simply be restored when necessary instead of the redundant computation of $\bar{P}_{n,m-2}(\sin\varphi)$ and $\bar{P}_{n,m+2}(\sin\varphi)$ in Eq. (79). The formulae for the remaining elements $T_{yy}, T_{zz}, T_{xy}, T_{xz}, T_{yz}$ may easily be modified in the same way.